

ON LINEARLY ORDERED  $H$ -CLOSED TOPOLOGICAL SEMILATTICES

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ABSTRACT. We give a criterium when a linearly ordered topological semilattice is  $H$ -closed. We also prove that any linearly ordered  $H$ -closed topological semilattice is absolutely  $H$ -closed and we show that every linearly ordered semilattice is a dense subsemilattice of an  $H$ -closed topological semilattice.

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [2]–[5]. If  $A$  is a subset of a topological space  $X$ , then we shall denote the closure of the set  $A$  in  $X$  by  $\text{cl}_X(A)$ . We shall denote the first infinite cardinal by  $\omega$ .

A *semilattice* is a semigroup with a commutative idempotent semigroup operation. If  $S$  is a topological space equipped with a continuous semigroup operation, then  $S$  is called a *topological semigroup*. A *topological semilattice* is a topological semigroup which is algebraically a semilattice.

If  $E$  is a semilattice, then the semilattice operation on  $E$  determines the partial order  $\leq$  on  $E$ :

$$e \leq f \quad \text{if and only if} \quad ef = fe = e.$$

This order is called *natural*. An element  $e$  of a semilattice  $E$  is called *minimal* (*maximal*) if  $f \leq e$  ( $e \leq f$ ) implies  $f = e$  for  $f \in E$ . For elements  $e$  and  $f$  of a semilattice  $E$  we write  $e < f$  if  $e \leq f$  and  $e \neq f$ . A semilattice  $E$  is said to be *linearly ordered* or a *chain* if the natural order on  $E$  is linear.

Let  $S$  be a semilattice and  $e \in S$ . We denote  $\downarrow e = \{f \in S \mid f \leq e\}$  and  $\uparrow e = \{f \in S \mid e \leq f\}$ . If  $S$  is a topological semilattice then Propositions VI.1.6(ii) and VI.1.14 of [5] imply that  $\uparrow e$  and  $\downarrow e$  are closed subsets in  $S$  for any  $e \in S$ .

Let  $E$  be a linearly ordered topological semilattice. Since  $\downarrow e$  and  $\uparrow e$  are closed for each  $e \in E$ , it follows that the topology of  $E$  refines the order topology. Thus the principal objects which we shall consider can be alternatively viewed as linearly ordered sets equipped with the semilattice operation of taking minimum and equipped with a topology refining the order topology for which the semilattice operation is continuous.

Let  $\mathcal{S}$  be some class of topological semigroups. A semigroup  $S \in \mathcal{S}$  is called  *$H$ -closed in  $\mathcal{S}$* , if  $S$  is a closed subsemigroup of any topological semigroup  $T \in \mathcal{S}$  which contains  $S$  both as a subsemigroup and as a topological space. If  $\mathcal{S}$  coincides with the class of all topological semigroups, then the semigroup  $S$  is called  *$H$ -closed*. The  $H$ -closed topological semigroups were introduced by Stepp [8], they were called *maximal semigroups*. A topological semigroup  $S \in \mathcal{S}$  is called *absolutely  $H$ -closed in the class  $\mathcal{S}$* , if any continuous homomorphic image of  $S$  into  $T \in \mathcal{S}$  is  $H$ -closed in  $\mathcal{S}$ . If  $\mathcal{S}$  coincides with the class of all topological semigroups, then the semigroup  $S$  is called *absolutely  $H$ -closed*.

An algebraic semigroup  $S$  is called *algebraically  $h$ -closed in  $\mathcal{S}$* , if  $S$  equipped with discrete topology  $\mathfrak{d}$  is absolutely  $H$ -closed in  $\mathcal{S}$  and  $(S, \mathfrak{d}) \in \mathcal{S}$ . If  $\mathcal{S}$  coincides with the class of all topological semigroups, then the semigroup  $S$  is called *algebraically  $h$ -closed*. Absolutely  $H$ -closed topological semigroups and algebraically  $h$ -closed semigroups were introduced by Stepp in [9], they were called *absolutely maximal* and *algebraic maximal*, respectively. Gutik and Pavlyk [6] observed that a topological semilattice is [absolutely]  $H$ -closed if and only if it is [absolutely]  $H$ -closed in the class of topological semilattices.

Stepp [9] proved that a semilattice  $E$  is algebraically  $h$ -closed if and only if any maximal chain in  $E$  is finite and he asked the following question: *Is every  $H$ -closed topological semilattice absolutely  $H$ -closed?* In the present paper we give a criterium when a linearly ordered topological semilattice is  $H$ -closed. We also prove that every linearly ordered  $H$ -closed topological semilattice is absolutely  $H$ -closed, we show that every linearly ordered semilattice is a dense subsemilattice of an  $H$ -closed topological semilattice,

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and we give an example of a linearly ordered  $H$ -closed locally compact topological semilattice which does not embed into a compact topological semilattice.

Let  $C$  be a maximal chain of a topological semilattice  $E$ . Then  $C = \bigcap_{e \in C} (\downarrow e \cup \uparrow e)$ , and hence  $C$  is a closed subsemilattice of  $E$ . Therefore we obtain the following:

**Lemma 1.** Let  $L$  be a linearly ordered subsemilattice of a topological semilattice  $E$ . Then  $\text{cl}_E(L)$  is a linearly ordered subsemilattice of  $E$ .

A linearly ordered topological semilattice  $E$  is called *complete* if every non-empty subset of  $S$  has inf and sup.

**Theorem 2.** A linearly ordered topological semilattice  $E$  is  $H$ -closed if and only if the following conditions hold:

- (i)  $E$  is complete;
- (ii)  $x = \sup A$  for  $A = \downarrow A \setminus \{x\}$  implies  $x \in \text{cl}_E A$ , whenever  $A \neq \emptyset$ ; and
- (iii)  $x = \inf B$  for  $B = \uparrow B \setminus \{x\}$  implies  $x \in \text{cl}_E B$ , whenever  $B \neq \emptyset$

*Proof.* ( $\Leftarrow$ ) Suppose to the contrary that there exists a linearly ordered non- $H$ -closed topological semilattice  $E$  which satisfies the conditions (i), (ii), and (iii). Then by Lemma 1 there exists a linearly ordered topological semilattice  $T$  such that  $E$  is a dense proper subsemilattice of  $T$ . Let  $x \in T \setminus E$ . Condition (i) implies that  $x \neq \sup E$  and  $x \neq \inf E$ , otherwise  $E$  is not complete. Therefore we have  $\inf E < x < \sup E$ . Let  $A(x) = (T \setminus \uparrow x) \cap E$  and  $B(x) = (T \setminus \downarrow x) \cap E$ . Since  $E$  is complete, we have  $\sup A(x) \in E$  and  $\inf B(x) \in E$ . Let  $s = \sup A(x)$  and  $i = \inf B(x)$ . We observe that  $s < x < i$ , otherwise, if  $x < s$ , then  $A = \downarrow x \cap E = (\downarrow s \setminus \{s\})_E$  is a closed subset in  $E$ , which contradicts the condition (ii), and if  $i < x$ , then  $B = \uparrow x \cap E = (\uparrow i \setminus \{i\})_E$  is a closed subset of  $E$ , which contradicts the condition (iii). Then  $\uparrow i$  and  $\downarrow s$  are closed subsets of  $T$  and  $T = \downarrow s \cup \uparrow i \cup \{x\}$ . Therefore  $x$  is an isolated point of  $T$ . This contradicts the assumption that  $x \in T \setminus E$  and  $E$  is a dense subspace of  $T$ . This contradiction implies that  $E$  is an  $H$ -closed topological semilattice.

( $\Rightarrow$ ) At first we shall show that  $\sup A \in E$  for every infinite subset  $S$  of  $E$ . Suppose to the contrary, that there exists an infinite subset  $A$  in  $E$  such that  $A$  has no sup in  $E$ . Since  $S$  is a linearly ordered semilattice, we have that the set  $\downarrow A$  also has no sup in  $E$ . We consider two cases:

- (a)  $E \setminus \downarrow A \neq \emptyset$ ; and
- (b)  $E \setminus \downarrow A = \emptyset$ .

In case (a) the set  $B = E \setminus \downarrow A$  has no inf, since  $E$  is a linearly ordered semilattice, otherwise  $\inf B = \sup A \in E$ .

Let  $x \notin E$ . We put  $E^* = E \cup \{x\}$ . We extend the semilattice operation from  $E$  onto  $E^*$  as follows:

$$x \cdot y = y \cdot x = \begin{cases} x, & \text{if } y \in E \setminus \downarrow A; \\ x, & \text{if } y = x; \\ y, & \text{if } y \in \downarrow A. \end{cases}$$

Obviously, the semilattice operation on  $E^*$  determines a linear order on  $E^*$ .

We define a topology  $\tau^*$  on  $E^*$  as follows. Let  $\tau$  be the topology on  $E$ . At any point  $a \in E = E^* \setminus \{x\}$  bases of topologies  $\tau^*$  and  $\tau$  coincide. We put

$$\mathcal{B}^*(x) = \{V_b^a(x) = E \setminus (\downarrow b \cup \uparrow a) \mid a \in E \setminus \downarrow A, b \in \downarrow A\}.$$

Since the set  $E \setminus \downarrow A$  has no inf and the set  $\downarrow A$  has no sup, the conditions (BP1)—(BP3) of [4] hold for the family  $\mathcal{B}^*(x)$  and  $\mathcal{B}^*(x)$  is a base of a Hausdorff topology  $\tau^*$  at the point  $x \in E^*$ .

Let  $c \in E \setminus \downarrow A$  and  $d \in \downarrow A$ . Then there exist  $a \in E \setminus \downarrow A$  and  $b \in \downarrow A$  such that  $d < b < x < a < c$ . Then for any open neighbourhoods  $V(c)$  and  $V(d)$  of the points  $c$  and  $d$ , respectively, such that  $V(c) \subseteq E \setminus \downarrow a = E^* \setminus \downarrow a$  and  $V(d) \subseteq E \setminus \uparrow b = E^* \setminus \uparrow b$  we have

$$V_b^a(x) \cdot V(d) \subseteq V(d) \quad \text{and} \quad V_b^a(x) \cdot V(c) \subseteq V_b^a(x).$$

We also have  $V_b^a(x) \cdot V_b^a(x) \subseteq V_b^a(x)$  for all  $a \in E \setminus \downarrow A$  and  $b \in \downarrow A$ . Therefore  $(E^*, \tau^*)$  is a Hausdorff topological semilattice which contains  $E$  as a dense non-closed subsemilattice. This contradicts the assumption that  $E$  is an  $H$ -closed topological semilattice.

Consider case (b). Let  $y \notin E$ . We put  $E^* = E \cup \{y\}$  and extend the semilattice operation from  $E$  onto  $E^*$  as follows:

$$y \cdot s = s \cdot y = \begin{cases} y, & \text{if } s = y; \\ s, & \text{if } s \neq y. \end{cases}$$

Obviously, the semilattice operation on  $E^*$  determines a linear order on  $E^*$ .

We define a topology  $\tau^*$  on  $E^*$  as follows. Let  $\tau$  be the topology on  $E$ . At any point  $a \in E = E^* \setminus \{y\}$  bases of topologies  $\tau^*$  and  $\tau$  coincide. We put

$$\mathcal{B}^*(x) = \{V_b(x) = E \setminus \downarrow b \mid b \in \downarrow A = E\}.$$

Since the set  $E = \downarrow E$  has no sup, the conditions (BP1)—(BP3) of [4] hold for the family  $\mathcal{B}^*(x)$  and  $\mathcal{B}^*(x)$  is a base of a Hausdorff topology  $\tau^*$  at the point  $x \in E^*$ .

Let  $c \in E$ . Then there exists  $b \in E$  such that  $c < b < y$  and for any open neighbourhood  $V_b(y)$  of  $y$  and any open neighbourhood  $V(c)$  such that  $V(c) \subseteq E \setminus \uparrow b$  we have  $V_b(y) \cdot V(c) \subseteq V(c)$ . We also have  $V_b(y) \cdot V_b(y) \subseteq V_b(y)$  for all  $b \in \downarrow A = E$ . Therefore  $(E^*, \tau^*)$  is a Hausdorff topological semilattice which contains  $E$  as a dense non-closed subsemilattice. This contradicts the assumption that  $E$  is an  $H$ -closed topological semilattice. The obtained contradictions imply that every subset of the semilattice  $E$  has sup. The proof of the fact that every subset of  $E$  has an inf is similar.

Next we show that for every  $H$ -closed linearly ordered topological semilattice  $E$  condition (ii) holds. Suppose that there exists  $x \in E$  such that  $x = \sup(\downarrow x \setminus \{x\})$  and  $x \notin \text{cl}_E(\downarrow x \setminus \{x\})$ . Since the topological semilattice  $E$  is linearly ordered,  $L^\circ(x) = \downarrow x \setminus \{x\}$  is a clopen subset of  $E$ .

Let  $g \notin E$ . We extend the semilattice operation from  $E$  onto  $E^\circ = E \cup \{g\}$  as follows:

$$g \cdot s = s \cdot g = \begin{cases} g, & \text{if } s \in \uparrow x; \\ s, & \text{if } s \in L^\circ(x). \end{cases}$$

Obviously, the semilattice operation on  $E^\circ$  determines a linear order on  $E^\circ$ .

We define a topology  $\tau^\circ$  on  $E^\circ$  as follows. Let  $\tau$  be the topology on  $E$ . At any point  $a \in E = E^\circ \setminus \{g\}$  bases of topologies  $\tau^\circ$  and  $\tau$  coincide. We put

$$\mathcal{B}^\circ(g) = \{U_s(g) = \{g\} \cup L^\circ(x) \setminus \downarrow s \mid s \in L^\circ(x)\}.$$

Since  $\sup L^\circ(x) = x$ , the set  $U_s(g)$  is non-singleton for any  $s \in L^\circ(x)$ . Therefore the conditions (BP1)—(BP3) of [4] hold for the family  $\mathcal{B}^\circ(g)$  and  $\mathcal{B}^\circ(g)$  is a base of the topology  $\tau^\circ$  at the point  $g \in E^\circ$ . Also since the set  $\uparrow x$  is closed in  $(S^\circ, \tau^\circ)$  and  $\sup L^\circ(x) = x$ , we have that the topology  $\tau^\circ$  is Hausdorff. The proof of the continuity of the semilattice operation in  $(S^\circ, \tau^\circ)$  is similar as for  $(S^*, \tau^*)$  and  $(S^*, \tau^*)$ . Thus condition (ii) holds.

The proof of the assertion that if  $E$  is a linearly ordered  $H$ -closed topological semilattice, then the condition (iii) holds, is similar. Therefore the proof of the theorem is complete.  $\square$

Since the conditions (i)—(iii) of Theorem 2 are preserved by continuous homomorphisms, we have the following:

**Theorem 3.** *Every linearly ordered  $H$ -closed topological semilattice is absolutely  $H$ -closed.*

Theorem 2 also implies the following:

**Corollary 4.** *Every linearly ordered  $H$ -closed topological semilattice contains maximal and minimal idempotents.*

Theorems 2 and 3 imply the following:

**Corollary 5.** *Let  $E$  be a linearly ordered  $H$ -closed topological semilattice and  $e \in E$ . Then  $\uparrow e$  and  $\downarrow e$  are (absolutely)  $H$ -closed topological semilattices.*

**Theorem 6.** *Every linearly ordered topological semilattice is a dense subsemilattice of an  $H$ -closed linearly ordered topological semilattice.*

*Proof.* Let  $E$  be a linearly ordered topological semilattice and let  $E_a$  be an algebraic copy of  $E$ . Then  $E_a$  with operation  $\inf$  and  $\sup$  is a lattice. It is well known that every lattice embeds into a complete lattice (cf. [2, Theorem V.2.1]). In our construction we shall use the idea of proofs of Theorem V.2.1 and Lemma V.2.1 in [2]. We denote the lattice of all ideals of  $E_a$  by  $\tilde{E}_a$ . Then pointwise operations  $\inf$  and

sup on  $\tilde{E}_a$  coincide with  $\bigcap$  and  $\bigcup$  on  $\tilde{E}_a$ , respectively. Since  $E_a$  is a linearly ordered semilattice, we can identify  $E_a$  with the subsemilattice of all principal ideals of  $E_a$  in  $\tilde{E}_a$ .

For an ideal  $I \in \tilde{E}_a$  and a principal ideal  $I_e \in \tilde{E}_a$  generated by an idempotent  $e$  we put  $I_e \rho I$  if and only if every open neighbourhood of  $e$  intersects  $I$ . Since  $E_a$  and  $\tilde{E}_a$  are linearly ordered semilattices, for principal ideals  $I_e$  and  $I_f$  generated by idempotents  $e$  and  $f$  from  $E_a$ , respectively, we have  $I_e \rho I_f$  if and only if  $I_e = I_f$ , i.e.  $e = f$ . We put  $\alpha = \Delta \cap \rho \cup \rho^{-1}$ . Obviously the relation  $\alpha$  is an equivalence on  $\tilde{E}_a$ . Since  $\tilde{E}_a$  is a linearly ordered semilattice,  $\alpha$  is a congruence on  $\tilde{E}_a$  and hence  $\tilde{E} = \tilde{E}_a / \alpha$  is also a linearly ordered semilattice. We observe that  $E_a$  is a subsemilattice of  $\tilde{E}$ .

We define a topology  $\tilde{\tau}$  on  $\tilde{E}$  as follows. Let  $\tau$  be the topology on  $E$ . At any point  $a \in E_a \subset \tilde{E}$  bases of topologies  $\tilde{\tau}$  and  $\tau$  coincide. For  $x \in \tilde{E} \setminus E_a$  we put

$$\tilde{\mathcal{B}}(x) = \{V_b(x) = \downarrow x \setminus \downarrow b \mid b \in \downarrow x \cap E_a\}.$$

Then conditions (BP1)—(BP3) of [4] hold for the family  $\tilde{\mathcal{B}}(x)$  and  $\tilde{\mathcal{B}}(x)$  is a base of a topology  $\tilde{\tau}$  at the point  $x$  and since  $\tau$  is Hausdorff, so is  $\tilde{\tau}$ . We also observe that the definition of  $\tilde{\tau}$  implies that  $\downarrow e$  and  $\uparrow e$  are closed subset of the topological space  $(\tilde{E}, \tilde{\tau})$ . Obviously the semilattice operation on  $(\tilde{E}, \tilde{\tau})$  is continuous,  $(\tilde{E}, \tilde{\tau})$  satisfies the conditions (i) and (ii) of Theorem 2 and  $E$  is a dense subsemilattice of  $(\tilde{E}, \tilde{\tau})$ .

We denote the lattice of all filters of  $\tilde{E}$  by  $\mathcal{F}(E)$ . Then pointwise operations inf and sup on  $\mathcal{F}(E)$  coincide with  $\bigcup$  and  $\bigcap$  on  $\mathcal{F}(E)$ , respectively. Since  $\tilde{E}$  is a linearly ordered semilattice, we can identify  $\tilde{E}$  with the subsemilattice in  $\mathcal{F}(E)$  of all principal filters of  $\tilde{E}$ . By the dual theorem to Theorem V.2.1 of [2] the lattice  $\mathcal{F}(E)$  is complete, and since  $\tilde{E}$  is linearly ordered, so is  $\mathcal{F}(E)$ .

For a filter  $F \in \tilde{E}_a$  and a principal filter  $F_e \in \mathcal{F}(E)$  generated by an idempotent  $e \in \tilde{E}$  we put  $F_e \tilde{\rho} F$  if and only if every open neighbourhood of  $e$  intersects  $F$ . Since  $\tilde{E}$  and  $\mathcal{F}(E)$  are linearly ordered semilattices, for principal filters  $F_e$  and  $F_f$  generated by idempotents  $e$  and  $f$  from  $\tilde{E}$ , respectively, we have  $F_e \tilde{\rho} F_f$  if and only if  $F_e = F_f$ , i.e.  $e = f$ . We put  $\tilde{\alpha} = \Delta \cap \tilde{\rho} \cup \tilde{\rho}^{-1}$ . Obviously  $\tilde{\alpha}$  is an equivalence on  $\mathcal{F}(E)$ . Since  $\mathcal{F}(E)$  is a linearly ordered semilattice,  $\tilde{\alpha}$  is a congruence on  $\mathcal{F}(E)$  and hence  $\tilde{\mathcal{F}}(E) = \mathcal{F}(E) / \tilde{\alpha}$  is a linearly ordered semilattice.

We define a topology  $\tau_{\mathcal{F}}$  on  $\tilde{\mathcal{F}}(E)$  as follows. At any point  $a \in \tilde{E} \subseteq \tilde{\mathcal{F}}(E)$  bases of topologies  $\tau_{\mathcal{F}}$  and  $\tilde{\tau}$  coincide. For  $x \in \tilde{\mathcal{F}}(E) \setminus \tilde{E}$  we put

$$\tilde{\mathcal{B}}_{\mathcal{F}}(x) = \{W^b(x) = \uparrow x \setminus \uparrow b \mid b \in \uparrow x \cap \tilde{E}\}.$$

Then conditions (BP1)—(BP3) of [4] hold for the family  $\tilde{\mathcal{B}}_{\mathcal{F}}(x)$  and  $\tilde{\mathcal{B}}_{\mathcal{F}}(x)$  is a base of a topology  $\tau_{\mathcal{F}}$  at the point  $x$  and since  $\tilde{\tau}$  is Hausdorff, so is  $\tau_{\mathcal{F}}$ . Also we observe that the definition of  $\tau_{\mathcal{F}}$  implies that  $\downarrow e$  and  $\uparrow e$  are closed subsets of the topological space  $(\tilde{\mathcal{F}}(E), \tau_{\mathcal{F}})$ , and hence the semilattice operation on  $(\tilde{\mathcal{F}}(E), \tau_{\mathcal{F}})$  is continuous,  $(\tilde{E}, \tilde{\tau})$  satisfies the conditions (i) and (iii) of Theorem 2 and  $E$  is a dense subsemilattice of  $(\tilde{\mathcal{F}}(E), \tau_{\mathcal{F}})$ .

Further we shall show that the condition (ii) of Theorem 2 holds for the topological semilattice  $(\tilde{\mathcal{F}}(E), \tau_{\mathcal{F}})$ . Suppose to the contrary that there exists a lower subset  $A$  of  $\mathcal{F}(E)$  such that  $\sup A = x \notin A$  and  $x \notin \text{cl}_{\mathcal{F}(E)} A$ . Then  $A = \mathcal{F}(E) \setminus \uparrow x$  and  $A$  are clopen subsets of the topological space  $(\mathcal{F}(E), \tau_{\mathcal{F}})$ . Since  $x = \sup A \notin A$ , there exists an increasing family of ideals  $\mathcal{I} = \{I_\alpha \mid \alpha \in \mathcal{A}\}$  of the semilattice  $E_a$  such that  $I_\alpha \subset I_\beta$  whenever  $\alpha < \beta$ ,  $\alpha, \beta \in \mathcal{A}$ ,  $\sup \bigcup \mathcal{I} = x$ , and  $\bigcup \mathcal{I} \subset A$ . The existence of the family  $\mathcal{I}$  follows from the fact that the semilattice  $\tilde{E}$  is a dense subsemilattice of  $(\tilde{\mathcal{F}}(E), \tau_{\mathcal{F}})$ . However,  $\bigcup \mathcal{I}$  is an ideal in  $E_a$  and hence  $\sup \bigcup \mathcal{I} \in A$ , a contradiction. The obtained contradiction implies that statement (ii) holds for the topological semilattice  $(\tilde{\mathcal{F}}(E), \tau_{\mathcal{F}})$ .  $\square$

**Example 7.** Let  $\mathbb{N}$  be the set of positive integers. Let  $\{x_n\}$  be an increasing sequence in  $\mathbb{N}$ . Put  $\mathbb{N}^* = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ . We define the semilattice operation on  $\mathbb{N}^*$  as follows  $ab = \min\{a, b\}$ , for  $a, b \in \mathbb{N}^*$ . Obviously, 0 is the zero element of  $\mathbb{N}^*$ . We put  $U_n(0) = \{0\} \cup \{\frac{1}{x_k} \mid k \geq n\}$ ,  $n \in \mathbb{N}$ . A topology  $\tau$  on  $\mathbb{N}^*$  is defined as follows: all nonzero elements of  $\mathbb{N}^*$  are isolated points in  $\mathbb{N}^*$  and  $\mathcal{B}(0) = \{U_n(0) \mid n \in \mathbb{N}\}$  is the base of the topology  $\tau$  at the point  $0 \in \mathbb{N}^*$ . It is easy to see that  $(\mathbb{N}^*, \tau)$  is a countable linearly ordered  $\sigma$ -compact 0-dimensional scattered locally compact metrizable topological

semilattice and if  $x_{k+1} > x_k + 1$  for every  $k \in \mathbb{N}$ , then  $(\mathbb{N}^*, \tau)$  is a non-compact semilattice. We also observe that the family  $\text{Hom}(E, \{0, 1\})$  of all homomorphisms from a topological semilattice  $E$  into the discrete semilattice  $(\{0, 1\}, \min)$  separates points for the topological semilattice  $E$ .

Theorem 2 implies the following:

**Proposition 8.**  $(\mathbb{N}^*, \tau)$  is an  $H$ -closed topological semilattice.

**Remark 9.** Example 7 implies negative answers to the following questions:

- (1) Is every closed subsemilattice of an  $H$ -closed topological semilattice  $H$ -closed?
- (2) (I. Guran) Does every locally compact topological semilattice embed into a compact semilattice?
- (3) (cf. [1]) Does every globally bounded topological inverse Clifford semigroup embed into a compact semigroup?
- (4) Does every locally compact topological semilattice have a base with open order convex subsets?

**Remark 10.** Theorem 3 and Example 7 imply that a closed subsemilattice of an absolutely  $H$ -closed topological semilattice is not  $H$ -closed.

Example 11 implies that there exist topologically isomorphic linearly ordered topological semilattices  $E_1$  and  $E_2$  which is dense subsemilattice of linearly ordered topological semilattices  $S_1$  and  $S_2$ , respectively, such that  $S_1$  and  $S_2$  are not algebraically isomorphic.

**Example 11.** Let  $\mathbb{N}$  be the set of positive integers. Let  $S_1 = \{-\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$  and  $S_2 = \{-1 - \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{-1\} \cup \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$  with usual topology and operation  $\min$ . Then  $E_1 = \{-\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$  and  $E_2 = \{-1 - \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$  is discrete isomorphic semilattice, but the semilattices  $S_1$  and  $S_2$  are not algebraically isomorphic.

Theorem 12 gives a method of constructing new  $H$ -closed and absolutely  $H$ -closed topological semilattices from old.

**Theorem 12.** Let  $S = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$  be a topological semilattice such that:

- (i)  $S_\alpha$  is an (absolutely)  $H$ -closed topological semilattice for any  $\alpha \in \mathcal{A}$ ; and
- (ii) there exists an (absolutely)  $H$ -closed topological semilattice  $T$  such that  $T \subseteq S$  and  $S_\alpha S_\beta \subseteq T$  for all  $\alpha \neq \beta$ ,  $\alpha, \beta \in \mathcal{A}$ .

Then  $S$  is an (absolutely)  $H$ -closed topological semilattice.

*Proof.* We shall consider only the case when  $S$  is an absolutely  $H$ -closed topological semilattice. The proof in the other case is similar. Let  $h: S \rightarrow G$  be a continuous homomorphism from  $S$  into a topological semilattice  $G$ . Without loss of generality we may assume that  $\text{cl}_G(h(S)) = G$ .

Suppose that  $G \setminus h(S) \neq \emptyset$ . We fix  $x \in G \setminus h(S)$ . The absolute  $H$ -closedness of the topological semilattice  $T$  implies that there exists an open neighbourhood  $U(x)$  of the point  $x$  in  $G$  such that  $U(x) \cap h(T) = \emptyset$ . Since  $G$  is a topological semilattice, there exists an open neighbourhood  $V(x)$  of  $x$  in  $G$  such that  $V(x)V(x) \subseteq U(x)$ . Since the topological semilattice  $S_\alpha$  is absolutely  $H$ -closed, the neighbourhood  $V(x)$  intersects infinitely many semilattices  $h(S_\beta)$ ,  $\beta \in \mathcal{A}$ . Therefore  $V(x)V(x) \cap h(T) \neq \emptyset$ . This is in disagreement with the choice of the neighbourhood  $U(x)$ . This contradiction implies the assertion of the theorem.  $\square$

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